

# Bifurcation curves of subharmonic solutions

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## Abstract

We revisit a problem considered by Chow and Hale on the existence of subharmonic solutions for perturbed systems. In the analytic setting, under more general (weaker) conditions, we prove their results on the existence of bifurcation curves from the nonexistence to the existence of subharmonic solutions. In particular our results apply also when one has degeneracy to first order — i.e. when the subharmonic Melnikov function vanishes identically. Moreover we can deal as well with the case in which degeneracy persists to arbitrarily high orders, in the sense that suitable generalisations to higher orders of the subharmonic Melnikov function are also identically zero. In general the bifurcation curves are not analytic, and even when they are smooth they can form cusps at the origin: we say in this case that the curves are degenerate as the corresponding tangent lines coincide. The technique we use is completely different from that of Chow and Hale, and it is essentially based on rigorous perturbation theory.

## 1 Introduction

Subharmonic bifurcations have been extensively studied in the literature [11, 24]. The problem can be formulated as follows. Consider a two-dimensional autonomous system, and suppose that it has a periodic orbit of period  $T = 2\pi p/q$ , where  $p, q$  are relatively prime integers. Then one can be interested in studying whether, under the action of a periodic perturbation of period  $2\pi$ , some periodic solutions exist. Solutions with this property are called subharmonic solutions of order  $q/p$ .

Assume also that the perturbation depends on two parameters. A typical situation is when dissipation is present in the system [25, 28]; in this case two parameters naturally arise: the magnitude of the perturbation and the damping coefficient. An interesting problem is then to study the region in the space of parameters where subharmonic solutions can occur and to determine the bifurcation curves, which divide the regions of existence and non-existence of these solutions. Such a problem has been considered for instance by Chow and Hale [11]. They found that, under suitable assumptions on the unperturbed system (essentially a local anisochronicity condition) and on the perturbation, the bifurcation curves exist, are smooth and intersect with distinct tangent lines at the origin. The condition on the perturbation, if one takes the magnitude of the perturbation as one of the parameters, can be formulated in terms of the so-called subharmonic Melnikov function [29, 24]. It requires in particular that this function depends explicitly on the initial phase  $t_0$  of the unperturbed periodic solutions which persist under perturbation.

In this paper we recover the same result by Chow and Hale, in the analytic setting, and we show that the condition on the perturbation can be weakened. In particular the subharmonic Melnikov function can be independent of  $t_0$ . As a consequence the bifurcation curves can be degenerate, in the sense that they can have the same tangent at the origin, so that they form a cusp at the origin. Moreover, in general, they are not smooth. However if some further assumption is made they turn out to be analytic.

In the case of dissipative systems in the presence of forcing, such as those studied by Hale and Táboas [25, 11], our result is significantly stronger as it requires no assumption at all on the periodic perturbation. In particular we find the following result. Given any one-dimensional anisochronous mechanical system perturbed by a periodic forcing of magnitude  $\varepsilon$  and in the presence of dissipation, there can be analytic subharmonic solutions of order  $q/p$  only if the dissipation coefficient  $\gamma$  is below a threshold value  $\gamma_0(q/p, \varepsilon)$ . Here we show that for any rational value  $p/q$  there is an integer exponent  $m = m(q/p) \in \mathbb{R}^*$  such that  $\gamma_0(q/p, \varepsilon) = O(\varepsilon^m)$ . This can be related, in a more general context, to a conjecture proposed in [1]. Moreover the case  $m(p/q) = \infty$  corresponds to infinitely many cancellations, one at each perturbation order, which makes such a case very unlikely. Therefore, up to these exceptional cases, we can say that any resonant torus with frequency commensurate with the frequency of the forcing term admits subharmonic solutions of the corresponding order. In other words, existence of any subharmonic solutions holds without making any assumption on the periodic perturbation, other than smoothness.

Our method is completely different from that of Chow and Hale. It is based on perturbation theory. More precisely we study the perturbation series of the subharmonic solutions: first we find conditions sufficient for these series to be well-defined to all orders, then we prove that if the perturbation is small enough convergence of the series can be proved. Technically, this is achieved by using the tree formalism, which has been originally introduced by Gallavotti [14], inspired by a pioneering paper by Eliasson [13], and thereafter has been applied in a long series of papers on KAM theory [6, 16, 17, 18, 22, 19, 20, 21]; see also [15] for a review. We note that with respect to these papers in our case the analysis is much easier as we deal with periodic solutions instead of quasi-periodic solutions. In this respect our analysis could be considered as a propaedeutic introduction to the tree formalism, in a case in which there is no small divisors problem, so that no multiscale analysis has to be introduced; see also [7, 8] for a similar situation. In particular Chow and Hale's assumptions on the perturbation reflect a case in which a first order analysis is enough to deduce existence of subharmonic solutions. By contrast our results allow the analysis of cases in which it can be necessary to go beyond the first order, in principle to arbitrarily high orders.

We also argue that in physical applications it can be essential to have a stronger result. Indeed, in a concrete example in which, for instance, the perturbation is a trigonometric polynomial, Chow and Hale's assumptions on the perturbation, even if they are generic, fail to be satisfied for most values of the periods  $T$ . For those values a first order condition is not sufficient to detect the existence of the subharmonic solution, and one must go to higher orders. The numerical simulations performed in [1] for a driven quartic oscillator in the presence of dissipation show that this is necessary if one wants to explain the numerical findings for some values of the parameters.

Our results should be compared also with [9, 10], where a different scenario, such as the persistence of the whole invariant manifold corresponding to the resonant torus, arises in a case in which the subharmonic Melnikov function vanishes identically. Our analysis shows that a situation of this kind is highly non-generic.

## 2 Main results

Consider the ordinary differential equation

$$\begin{cases} \dot{\alpha} = \omega(A) + \varepsilon F(\alpha, A, C, t), \\ \dot{A} = \varepsilon G(\alpha, A, C, t), \end{cases} \quad (2.1)$$

where  $(\alpha, A) \in \mathcal{M} := \mathbb{T} \times W$ , with  $W \subset \mathbb{R}$  an open set, the map  $A \rightarrow \omega(A)$  is analytic in  $A$ , and the functions  $F$  and  $G$  depend analytically on their arguments and are  $2\pi$ -periodic in  $\alpha$  and  $t$ . Finally,  $\varepsilon, C$  are two real parameters.

One could also introduce a further (analytic) dependence on  $\varepsilon$  in the functions  $F$  and  $G$ , and the

forthcoming analysis could be easily performed with some trivial adaptations. Therefore all the results and theorems stated below and in the next sections hold unchanged in that case too. Then, the formulation given in [11] is recovered, as a particular case, by introducing the parameter  $\gamma = \varepsilon C$ , and setting  $\mu = (\mu_1, \mu_2)$ , with  $\mu_1 = \varepsilon$  and  $\mu_2 = \gamma$ .

For  $\varepsilon = 0$  the variable  $A$  is kept fixed at some value  $A_0$ , while  $\alpha$  rotates with constant angular velocity  $\omega(A_0)$ . Hence the motion of the variables  $(\alpha, A, t)$  is quasi-periodic, and reduces to a periodic motion whenever  $\omega(A_0)$  becomes commensurate with 1. Define  $\alpha_0(t) = \omega(A_0)t$  and  $A_0(t) = A_0$ : in the *extended phase-space*  $\mathcal{M} \times \mathbb{R}$  the solution  $(\alpha_0(t), A_0(t), t + t_0)$  describes an invariant torus, which is uniquely determined by the “energy”  $A_0$ . If  $\omega(A_0)$  is rational we say that the torus is *resonant*. The parameter  $t_0$  will be called the *initial phase*: it fixes the initial datum on the torus.

As a particular case we can consider that  $(A, \alpha)$  are canonical coordinates (action-angle coordinates), but the formulation we are giving here is more general. In general all non-resonant tori are completely destroyed under perturbation, if no further hypotheses are made on the perturbations  $F, G$  (such as that the full system is Hamiltonian). Also the resonant tori disappear, but some remnants are left: indeed usually a finite number of periodic orbits, called *subharmonic solutions*, lying on the unperturbed torus, can survive under perturbation.

Denote by  $T_0(A) = 2\pi/\omega(A)$  the period of the trajectories on an unperturbed torus, and define  $\omega'(A) := d\omega(A)/dA$ . If  $\omega(A_0) = p/q \in \mathbb{Q}$ , call  $T = T(A_0) = 2\pi q$  the period of the trajectories in the extended phase space. We shall call  $q/p$  the *order* of the corresponding subharmonic solutions. Define

$$M(t_0, C) = \frac{1}{T} \int_0^T dt G(\alpha_0(t), A_0, C, t + t_0), \quad (2.2)$$

which is called the *subharmonic Melnikov function*. Here and in the following we do not write explicitly the dependence of the subharmonic Melnikov function on  $A_0$ , which is fixed once and for all. Note that  $M(t_0, C)$  is  $2\pi$ -periodic in  $t_0$ .

We make the following assumptions on the resonant torus with energy  $A_0$ .

**Hypothesis 1** *One has  $\omega'(A_0) \neq 0$ .*

**Hypothesis 2** *There exists an analytic curve  $t \rightarrow C_0(t)$  from  $[0, 2\pi)$  to  $\mathbb{R}$  such that  $M(t_0, C_0(t_0)) = 0$  and  $\partial M(t_0, C_0(t_0))/\partial C \neq 0$  for all  $t_0 \in [0, 2\pi)$ .*

The function  $C_0(t_0)$  is also  $2\pi$ -periodic in  $t_0$ . We prove the following result. We prefer to state the result in terms of the parameter  $\gamma = \varepsilon C$  — instead of  $C$  — to make more transparent the relation with [11].

**Theorem 1** *Consider the system (2.1) and assume that Hypotheses 1 and 2 hold for the resonant torus with energy  $A_0$  such that  $\omega(A_0) = p/q$ . There exist  $\varepsilon_0 > 0$  and two continuous functions  $\gamma_1(\varepsilon)$  and  $\gamma_2(\varepsilon)$ , with  $\gamma_1(0) = \gamma_2(0)$ ,  $\gamma_1(\varepsilon) \geq \gamma_2(\varepsilon)$  for  $\varepsilon \geq 0$  and  $\gamma_1(\varepsilon) \leq \gamma_2(\varepsilon)$  for  $\varepsilon \leq 0$ , such that (2.1) has at least one subharmonic solution of order  $q/p$  for  $\gamma_2(\varepsilon) \leq \varepsilon C \leq \gamma_1(\varepsilon)$  when  $\varepsilon \in (0, \varepsilon_0)$  and for  $\gamma_1(\varepsilon) \leq \varepsilon C \leq \gamma_2(\varepsilon)$  when  $\varepsilon \in (-\varepsilon_0, 0)$ .*

The situation is depicted in Figure 1, in a case in which the two functions  $\gamma_1$  and  $\gamma_2$  are smooth. The graphs described by the two functions are called the *bifurcation curves* of the subharmonic solutions: they divide the plane into two disjoint sets such that only in one of them there are analytic subharmonic solutions.

In general the functions  $\gamma_1$  and  $\gamma_2$  are not smooth. However, if some further assumptions are made on the subharmonic Melnikov function, smoothness (in fact analyticity) in  $\varepsilon$  can be obtained. Denote by  $C'_0(t_0)$  and  $C''_0(t_0)$  the first and second derivatives of the function  $C_0(t_0)$  with respect to  $t_0$ .

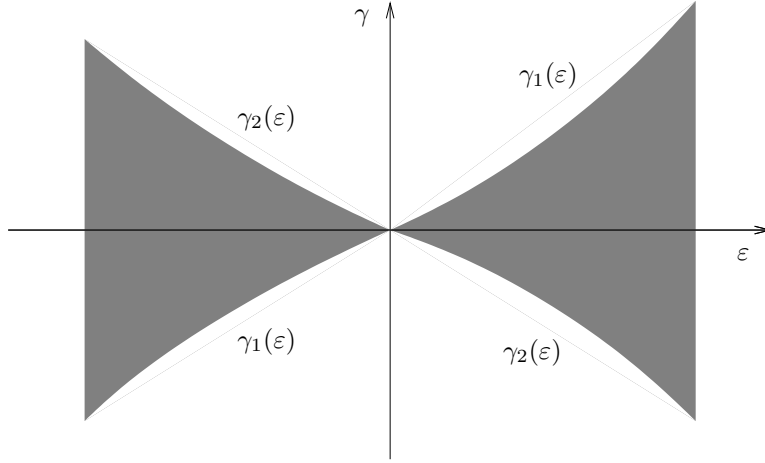


Figure 1: Set of existence (grey region) of subharmonic solutions in the plane  $(\varepsilon, \gamma)$ , in a case in which the two bifurcation curves  $\varepsilon \rightarrow \gamma_1(\varepsilon)$  and  $\varepsilon \rightarrow \gamma_2(\varepsilon)$  are smooth and have different tangent lines at the origin.

**Hypothesis 3** *If  $t_m$  and  $t_M$  are the values in  $[0, 2\pi)$  for which the function  $C_0(t_0)$  attains its minimum and its maximum, respectively, then  $C_0''(t_m)C_0''(t_M) \neq 0$ .*

The following result holds.

**Theorem 2** *Consider the system (2.1) and assume that Hypotheses 1, 2 and 3 hold for the resonant torus with energy  $A_0$  such that  $\omega(A_0) = p/q$ . There exist  $\varepsilon_0 > 0$  and two functions  $\gamma_1(\varepsilon)$  and  $\gamma_2(\varepsilon)$ , analytic for  $|\varepsilon| < \varepsilon_0$ , with  $\gamma_1(0) = \gamma_2(0)$ ,  $\gamma_1(\varepsilon) > \gamma_2(\varepsilon)$  for  $\varepsilon > 0$  and  $\gamma_1(\varepsilon) < \gamma_2(\varepsilon)$  for  $\varepsilon < 0$ , and with different tangent lines at the origin, such that (2.1) has at least one subharmonic solution of order  $q/p$  for  $\gamma_2(\varepsilon) \leq \varepsilon C \leq \gamma_1(\varepsilon)$  when  $\varepsilon \in (0, \varepsilon_0)$  and for  $\gamma_1(\varepsilon) \leq \varepsilon C \leq \gamma_2(\varepsilon)$  when  $\varepsilon \in (-\varepsilon_0, 0)$ .*

Theorem 2 is analogous to Theorem 2.1 of [11], Section 11 — in the analytic setting instead of the differentiable one — while Theorem 1 requires fewer hypotheses. In particular it applies when Chow and Hale's  $h_k(\alpha)$  function vanishes identically. In that case the graphs of the two functions  $\gamma_1$  and  $\gamma_2$  form a cusp at the origin: we refer to this situation as a case of *degenerate bifurcation curves*, see Figure 2. We shall also see in Section 4 that in fact, under weaker assumptions than those made in Hypothesis 3, we can find smoothness of the bifurcation curves, in the following sense: under suitable assumptions there exist two analytic functions  $\tilde{\gamma}_1(\varepsilon)$  and  $\tilde{\gamma}_2(\varepsilon)$  such that  $\gamma_1(\varepsilon) = \max\{\tilde{\gamma}_1(\varepsilon), \tilde{\gamma}_2(\varepsilon)\}$  and  $\gamma_2(\varepsilon) = \min\{\tilde{\gamma}_1(\varepsilon), \tilde{\gamma}_2(\varepsilon)\}$  for  $\varepsilon > 0$ , and  $\gamma_1(\varepsilon) = \min\{\tilde{\gamma}_1(\varepsilon), \tilde{\gamma}_2(\varepsilon)\}$  and  $\gamma_2(\varepsilon) = \max\{\tilde{\gamma}_1(\varepsilon), \tilde{\gamma}_2(\varepsilon)\}$  for  $\varepsilon < 0$ . We refer to Hypothesis 4 and Theorem 3 in Section 4 for a precise formulation of the results.

We shall see in Section 4 — cf. Theorem 4 — that for  $p = 1$  one has at least  $2q$  subharmonic solutions of order  $q$  as far as  $\min\{\gamma_1(\varepsilon), \gamma_2(\varepsilon)\} < \gamma < \max\{\gamma_1(\varepsilon), \gamma_2(\varepsilon)\}$  and at least  $q$  subharmonic solutions of order  $q$  when  $(\varepsilon, \gamma)$  belongs to one of the bifurcation curves, that is when either  $\gamma = \gamma_1(\varepsilon)$  or  $\gamma = \gamma_2(\varepsilon)$ . This agrees with Chow and Hale's Theorem 2.1 in [11] in the cases in which it applies.

Possible extensions of Chow and Hale's results could be looked for in another direction, such as that of relaxing the hypothesis on the unperturbed system. This problem has been studied, for instance, in [26, 27, 30].

The rest of the paper is organised as follows. Sections 3 and 4 are devoted to the proof of Theorems 1 and 2. More precisely, in Section 2 we show the existence of a subharmonic solution in the form of a formal power series, while in Section 4 we prove the convergence of the series, and we also state Theorem 3, which provides the aforementioned extensions of Theorem 2, and Theorem 4 on the minimal number

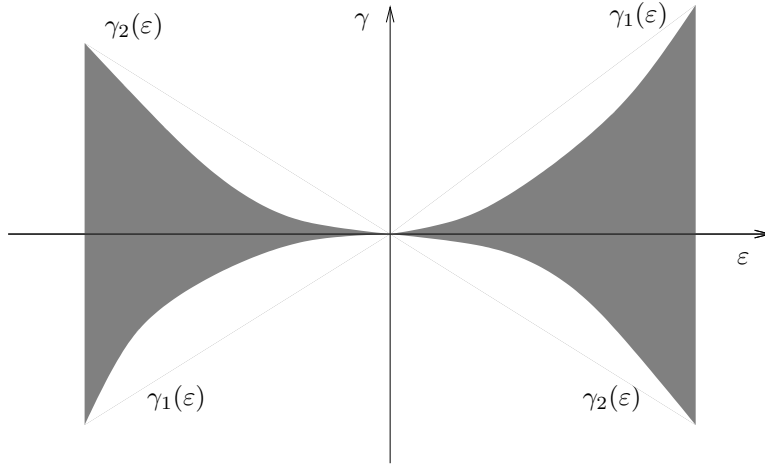


Figure 2: In general the bifurcation curves are not smooth. Even when they are smooth they can have the same tangent at the origin: in this case we say that the bifurcation curves are degenerate. The grey region in the figure represents a case in which the bifurcation curves are smooth and both of them have tangent lines parallel to the  $\varepsilon$ -axis.

of subharmonic solutions of order  $q$ . In Section 5 we discuss, as an application of our results, the case of a forced one-dimensional system in the presence of dissipation: this will lead to Theorems 5 and 6 which extend the results of Hale and Táboas [25].

Some final comments, and a comparison with the standard Melnikov theory [29, 24], are provided in Section 6. In particular we shall formulate two theorems: Theorem 7 corresponds to the result usually discussed in the literature [24], while Theorem 8 provides an extension to degenerate situations.

### 3 Existence of formal power series for the subharmonic solutions

We look for subharmonic solutions of (2.1) which are analytic in  $\varepsilon$ . First, we shall try to find solutions in the form of formal power series in  $\varepsilon$

$$\alpha(t) = \alpha(t, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \alpha^{(k)}(t), \quad A(t) = A(t; \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k A^{(k)}(t), \quad (3.1)$$

where  $\alpha^{(0)}(t) = \omega(A_0)t$  and  $A^{(0)}(t) = A_0$ , with  $\omega(A_0) = p/q$ , and the functions  $\alpha^{(k)}(t)$  and  $A^{(k)}(t)$ , periodic with period  $T = 2\pi p$  for all  $k \in \mathbb{N}$ , are to be determined. We shall see that this will be possible provided the parameter  $C$  is chosen as a function of  $\varepsilon$ , again in the form of a formal power series in  $\varepsilon$

$$C = C(\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k C^{(k)}. \quad (3.2)$$

Moreover both the solution  $(\alpha(t), A(t))$  and the constant  $C$  will be found to depend on the initial phase  $t_0$ : in particular one has  $C(\varepsilon) = C(\varepsilon, t_0)$  such that  $C(\varepsilon, t_0 + 2\pi) = C(\varepsilon, t_0)$  and  $C^{(0)} = C_0(t_0)$ , and, as we shall see, a sufficient condition for formal solvability to hold is that Hypotheses 1 and 2 are satisfied.

If we introduce the decompositions (3.1) and (3.2) into (2.1) and we denote with  $W(t)$  the Wronskian matrix for the unperturbed linearised system, we obtain (cf. [1] for similar computations)

$$\begin{pmatrix} \alpha^{(k)}(t) \\ A^{(k)}(t) \end{pmatrix} = W(t) \begin{pmatrix} \bar{\alpha}^{(k)} \\ \bar{A}^{(k)} \end{pmatrix} + W(t) \int_0^t d\tau W^{-1}(\tau) \begin{pmatrix} F^{(k-1)}(\tau) \\ G^{(k-1)}(\tau) \end{pmatrix}, \quad (3.3)$$

where  $(\bar{\alpha}^{(k)}, \bar{A}^{(k)})$  are corrections to the initial conditions, and

$$F^{(k)}(t) = [F(\alpha, A, C, t + t_0)]^{(k)} := \sum_{m=0}^{\infty} \sum_{\substack{r_1, r_2, r_3 \in \mathbb{Z}_+ \\ r_1 + r_2 + r_3 = m}} \frac{\partial_1^{r_1} \partial_2^{r_2} \partial_3^{r_3}}{r_1! r_2! r_3!} F(\alpha_0(t), A_0, C_0, t + t_0) \\ \sum_{k_1 + \dots + k_m = k} \alpha^{(k_1)}(t) \dots \alpha^{(k_{r_1})}(t) A^{(k_{r_1+1})}(t) \dots A^{(k_{r_1+r_2})}(t) C^{(k_{r_1+r_2+1})} \dots C^{(k_m)}, \quad (3.4)$$

with an analogous definition holding for  $G^{(k)}(t)$ . Here and henceforth, given a function of several arguments we are denoting by  $\partial_k$  the derivative with respect to the  $k$ -th argument; hence, given the function  $F(\alpha, A, C, t + t_0)$  we have  $\partial_1 F = \partial F / \partial \alpha$ ,  $\partial_2 F = \partial F / \partial A$ , and  $\partial_3 F = \partial F / \partial C$ . Note that by construction both  $F^{(k)}(t)$  and  $G^{(k)}(t)$  depend only on the coefficients  $\alpha^{(k')}(t)$ ,  $A^{(k')}(t)$  and  $C^{(k')}(t)$  with  $k' \leq k$ .

The Wronskian matrix appearing in (3.3) can be written as

$$W(t) = \begin{pmatrix} 1 & \omega'(A_0)t \\ 0 & 1 \end{pmatrix}. \quad (3.5)$$

By using (3.5) in (3.3) we have

$$\begin{cases} \alpha^{(k)}(t) = \bar{\alpha}^{(k)} + t \omega'(A_0) \bar{A}^{(k)} + \int_0^t d\tau F^{(k-1)}(\tau) + \omega'(A_0) \int_0^t d\tau \int_0^\tau d\tau' G^{(k-1)}(\tau'), \\ A^{(k)}(t) = \bar{A}^{(k)} + \int_0^t d\tau G^{(k-1)}(\tau). \end{cases} \quad (3.6)$$

We obtain a periodic solution of period  $T$  if, to any order  $k \in \mathbb{N}$ , one has

$$\langle G^{(k-1)} \rangle := \frac{1}{T} \int_0^T d\tau G^{(k-1)}(\tau) = 0 \quad (3.7)$$

and

$$\omega'(A_0) \bar{A}^{(k)} + \langle F^{(k-1)} \rangle + \omega'(A_0) \langle \mathcal{G}^{(k-1)} \rangle = 0, \quad \mathcal{G}^{(k-1)}(t) = \int_0^t d\tau G^{(k-1)}(\tau), \quad (3.8)$$

where, given any  $T$ -periodic function  $H$  we denote by  $\langle H \rangle$  its mean, as done in (3.7).

The parameters  $\bar{\alpha}^{(k)}$  are left undetermined, and we can fix them arbitrarily, as we have the initial phase  $t_0$  which is still a free parameter. For instance we can set  $\bar{\alpha}^{(k)} = 0$  for all  $k \in \mathbb{N}$  or else we can define  $\bar{\alpha}^{(k)} = \alpha_k(t_0)$  for  $k \in \mathbb{N}$ , with the constants  $\alpha_k(t_0)$  to be fixed in the way which turns out to be more convenient for computations: we shall see in the next section a reasonable choice.

Therefore, if equation (3.7) is satisfied, we have

$$\begin{cases} \alpha^{(k)}(t) = \int_0^t d\tau \left( F^{(k-1)}(\tau) - \langle F^{(k-1)} \rangle \right) + \omega'(A_0) \int_0^t d\tau \left( \mathcal{G}^{(k-1)}(\tau) - \langle \mathcal{G}^{(k-1)} \rangle \right), \\ A^{(k)}(t) = \bar{A}^{(k)} + \mathcal{G}^{(k-1)}(t), \end{cases} \quad (3.9)$$

with

$$\bar{A}^{(k)} = -\frac{\langle F^{(k-1)} \rangle}{\omega'(A_0)} - \langle \mathcal{G}^{(k-1)} \rangle = 0, \quad (3.10)$$

which is well-defined as  $\omega'(A_0) \neq 0$  by Hypothesis 1.

So, in order to prove the formal solvability of (2.1) we have to check whether it is possible to fix the parameter  $C$ , as a function of  $\varepsilon$  and  $t_0$ , in such a way that (3.7) follows for all  $k \geq 1$ .

For  $k = 1$  the condition (3.7) reads

$$\langle G^{(0)} \rangle = M(t_0, C) = 0, \quad (3.11)$$

and we can choose  $C = C_0(t_0)$  so that this holds: this is assured by Hypothesis 2.

To higher order  $k \geq 1$  we can write

$$G^{(k)}(\alpha(t), A(t), C, t + t_0) = \partial_3 G(\alpha_0(t), A_0, C_0, t + t_0) C^{(k)} + \Gamma^{(k)}(\alpha(t), A(t), C, t + t_0), \quad (3.12)$$

where the function  $\Gamma^{(k)}(\alpha(t), A(t), C, t + t_0)$  depends on the coefficients  $C^{(k')}$  of  $C$  with  $k' < k$  (and on the functions  $\alpha^{(k')}(t)$  and  $A^{(k')}(t)$  with  $k' \leq k$ , of course). In other words, in (3.12) we have extracted explicitly the only term depending on  $C^{(k)}$ . Moreover one has

$$\langle \partial_3 G(\alpha_0(\cdot), A_0, C_0, \cdot + t_0) \rangle = \frac{1}{T} \int_0^T dt \partial_3 G(\alpha_0(t), A_0, C_0, t + t_0) = \frac{\partial}{\partial C} M(t_0, C_0), \quad (3.13)$$

and by Hypothesis 2 one has  $D(t_0) := \partial M(t_0, C_0(t_0))/\partial C \neq 0$ , so that (3.7) is satisfied provided  $C^{(k)}$  is chosen as

$$C^{(k)} = -\frac{1}{D(t_0)} \langle \Gamma^{(k)}(\alpha(\cdot), A(\cdot), C, \cdot + t_0) \rangle \equiv C_k(t_0). \quad (3.14)$$

Therefore we conclude that if we set  $C_0 = C_0(t_0)$  and, for all  $k \geq 1$ , we choose  $\bar{\alpha}^{(k)} = \alpha_k(t_0)$ ,  $\bar{A}^{(k)}$  according to (3.10) and  $C^{(k)} = C_k(t_0)$  according to (3.14), we obtain that in the expansions (3.1) the coefficients  $\alpha^{(k)}(t)$  and  $A^{(k)}(t)$  are well-defined periodic functions of period  $T$ . Of course this does not settle the problem of convergence of the series (3.1) and (3.2). This will be discussed in the next Section.

## 4 Convergence of the series for the subharmonic solutions

Here we shall prove that the formal power series found in Section 3 converge for  $\varepsilon$  small enough, say for  $|\varepsilon| < \varepsilon_0$  for some  $\varepsilon_0 > 0$ . Then for fixed  $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$  we shall find the range allowed for  $C$  by computing the supremum and the infimum, for  $t_0 \in [0, 2\pi)$  of the function  $t_0 \rightarrow C(\varepsilon, t_0)$ . The bifurcation curves will be defined in terms of the function  $C(\varepsilon, t_0)$  — cf. (3.2) — as

$$\gamma_1(\varepsilon) = \varepsilon \sup_{t_0 \in [0, 2\pi)} C(\varepsilon, t_0), \quad \gamma_2(\varepsilon) = \varepsilon \inf_{t_0 \in [0, 2\pi)} C(\varepsilon, t_0). \quad (4.1)$$

In general the functions (4.1) are not smooth in  $\varepsilon$ . We shall return to this at the end of the section.

To prove convergence of the series (3.1) and (3.2) it is more convenient to work in Fourier space. First of all let us define  $\omega = 2\pi/T = 1/q$  (note that  $\omega \neq \omega(A_0)$ ) and expand

$$F(\alpha, A, C, t + t_0) = \sum_{\nu, \sigma \in \mathbb{Z}} e^{i\nu\alpha} e^{i\sigma(t+t_0)} F_{\nu, \sigma}(A, C), \quad (4.2)$$

so that we can write

$$\partial_1^{r_1} \partial_2^{r_2} \partial_3^{r_3} F(\omega(A_0)t, A_0, C_0(t_0), t + t_0) = \sum_{\nu \in \mathbb{Z}} e^{i\nu\omega t} \sum_{\substack{\nu_0, \sigma_0 \in \mathbb{Z} \\ \nu_0 p + \sigma_0 q = \nu}} e^{i\sigma t_0} (i\nu_0)^{r_1} \partial_2^{r_2} \partial_3^{r_3} F_{\nu_0, \sigma_0}(A_0, C_0(t_0)), \quad (4.3)$$

and an analogous expression can be obtained with the function  $G$  replacing  $F$ . By the analyticity assumption on the functions  $F$  and  $G$ , we have the bounds

$$\left| \frac{\partial_2^{r_2} \partial_3^{r_3}}{r_2! r_3!} F_{\nu_0, \sigma_0}(A_0, C_0(t_0)) \right| \leq P Q_1^{r_1} Q_2^{r_2} e^{-\kappa(|\nu_0| + |\sigma_0|)}, \\ \left| \frac{\partial_2^{r_2} \partial_3^{r_3}}{r_2! r_3!} G_{\nu_0, \sigma_0}(A_0, C_0(t_0)) \right| \leq P Q_1^{r_1} Q_2^{r_2} e^{-\kappa(|\nu_0| + |\sigma_0|)}, \quad (4.4)$$

for suitable positive constants  $P, Q_1, Q_2, \kappa$ .

Then, let us write in (3.1)

$$\alpha^{(k)}(t) = \sum_{\nu \in \mathbb{Z}} e^{i\nu\omega t} \alpha_\nu^{(k)}, \quad A^{(k)}(t) = \sum_{\nu \in \mathbb{Z}} e^{i\nu\omega t} A_\nu^{(k)}, \quad (4.5)$$

so that (3.9) becomes

$$\alpha_\nu^{(k)} = \frac{F_\nu^{(k-1)}}{i\omega\nu} + \omega'(A_0) \frac{G_\nu^{(k-1)}}{(i\omega\nu)^2}, \quad A_\nu^{(k)} = \frac{G_\nu^{(k-1)}}{i\omega\nu}, \quad (4.6)$$

for all  $\nu \neq 0$ , whereas for  $\nu = 0$  one has

$$\alpha_0^{(k)} = \alpha_k(t_0) - \sum_{\substack{\nu \in \mathbb{Z} \\ \nu \neq 0}} \frac{F_\nu^{(k-1)}}{i\omega\nu} - \omega'(A_0) \sum_{\substack{\nu \in \mathbb{Z} \\ \nu \neq 0}} \frac{G_\nu^{(k-1)}}{(i\omega\nu)^2}, \quad A_0^{(k)} = \bar{A}^{(k)} - \sum_{\substack{\nu \in \mathbb{Z} \\ \nu \neq 0}} \frac{G_\nu^{(k-1)}}{i\omega\nu} = -\frac{F_0^{(k-1)}}{\omega'(A_0)}, \quad (4.7)$$

with  $\alpha_k(t_0)$  so far arbitrary and  $\bar{A}^{(k)}$  given by (3.10). The Fourier coefficients  $F_\nu^{(k-1)}$  and  $G_\nu^{(k-1)}$  can be read from (3.4) and the analogous expression for  $G^{(k)}(t)$ . Hence one has

$$F_\nu^{(k)} = [F(\alpha, A, C, t + t_0)]_\nu^{(k)} = \sum_{m=0}^{\infty} \sum_{\substack{r_1, r_2, r_3 \in \mathbb{Z}_+ \\ r_1 + r_2 + r_3 = m}} \sum_{\substack{\nu_0, \sigma_0, \nu_1, \dots, \nu_{r_1+r_2} \in \mathbb{Z} \\ \nu_0 p + \sigma_0 q + \nu_1 + \dots + \nu_{r_1+r_2} = \nu}} \frac{(i\nu_0)^{r_1}}{r_1!} e^{i\sigma_0 t_0} \\ \frac{\partial_2^{r_2} \partial_3^{r_3}}{r_2! r_3!} F_{\nu_0}(A_0, C_0(t_0)) \sum_{k_1 + \dots + k_m = k} \alpha_{\nu_1}^{(k_1)} \dots \alpha_{\nu_{r_1}}^{(k_{r_1})} A_{\nu_{r_1+1}}^{(k_{r_1+1})} \dots A_{\nu_{r_1+r_2}}^{(k_{r_1+r_2})} C^{(k_{r_1+r_2+1})} \dots C^{(k_m)}, \quad (4.8)$$

and an analogous definition holds for  $G_\nu^{(k)}$ .

Furthermore one has

$$C^{(k)} = -\frac{1}{D(t_0)} \Gamma_0^{(k)} \quad (4.9)$$

where

$$\Gamma_0^{(k)} = [\Gamma(\alpha, A, C, t + t_0)]_0^{(k)} = \sum_{m=0}^{\infty} \sum_{\substack{r_1, r_2, r_3 \in \mathbb{Z}_+ \\ r_1 + r_2 + r_3 = m}}^* \sum_{\substack{\nu_0, \sigma_0, \nu_1, \dots, \nu_{r_1+r_2} \in \mathbb{Z} \\ \nu_0 p + \sigma_0 q + \nu_1 + \dots + \nu_{r_1+r_2} = 0}} \frac{(i\nu_0)^{r_1}}{r_1!} e^{i\sigma_0 t_0} \\ \frac{\partial_2^{r_2} \partial_3^{r_3}}{r_2! r_3!} G_{\nu_0}(A_0, C_0(t_0)) \sum_{k_1 + \dots + k_m = k} \alpha_{\nu_1}^{(k_1)} \dots \alpha_{\nu_{r_1}}^{(k_{r_1})} A_{\nu_{r_1+1}}^{(k_{r_1+1})} \dots A_{\nu_{r_1+r_2}}^{(k_{r_1+r_2})} C^{(k_{r_1+r_2+1})} \dots C^{(k_m)}, \quad (4.10)$$

where  $*$  means that the term with  $r_1 = r_2 = 0$  and  $r_3 = 1$  has to be discarded — cf. (3.12).

Therefore we see from the first equation in (4.7) that it is convenient to fix

$$\alpha_k(t_0) = \sum_{\substack{\nu \in \mathbb{Z} \\ \nu \neq 0}} \frac{F_\nu^{(k-1)}}{i\omega\nu} + \omega'(A_0) \sum_{\substack{\nu \in \mathbb{Z} \\ \nu \neq 0}} \frac{G_\nu^{(k-1)}}{(i\omega\nu)^2} \implies \alpha_0^{(k)} = 0, \quad (4.11)$$

so that only the functions  $A^{(k)}(t)$  have the zeroth Fourier coefficient.

In particular for  $k = 1$  we find

$$\alpha_\nu^{(1)} = \frac{1}{i\omega\nu} \sum_{\substack{\nu_0, \sigma_0 \in \mathbb{Z} \\ \nu_0 p + \sigma_0 q = \nu}} e^{i\sigma_0 t_0} F_{\nu_0, \sigma_0}(A_0, C_0(t_0)) + \frac{\omega'(A_0)}{(i\omega\nu)^2} \sum_{\substack{\nu_0, \sigma_0 \in \mathbb{Z} \\ \nu_0 p + \sigma_0 q = \nu}} e^{i\sigma_0 t_0} G_{\nu_0, \sigma_0}(A_0, C_0(t_0)), \\ A_\nu^{(1)} = \frac{1}{i\omega\nu} \sum_{\substack{\nu_0, \sigma_0 \in \mathbb{Z} \\ \nu_0 p + \sigma_0 q = \nu}} e^{i\sigma_0 t_0} G_{\nu_0, \sigma_0}(A_0, C_0(t_0)), \quad (4.12)$$



for  $\nu \neq 0$ , and

$$A_0^{(1)} = -\frac{1}{\omega'(A_0)} \sum_{\substack{\nu_0, \sigma_0 \in \mathbb{Z} \\ \nu_0 p + \sigma_0 q = 0}} e^{i\sigma_0 t_0} F_{\nu_0, \sigma_0}(A_0, C_0(t_0)), \quad (4.13)$$

for  $\nu = 0$ , while by writing

$$\begin{aligned} C^{(1)} = & -\frac{1}{D(t_0)} \left( \sum_{\substack{\nu_1, \nu_2 \in \mathbb{Z} \\ \nu_1 + \nu_2 = 0}} \sum_{\substack{\nu_0, \sigma_0 \in \mathbb{Z} \\ \nu_0 p + \sigma_0 q = \nu_1}} e^{i\sigma_0 t_0} i\nu_0 G_{\nu_0, \sigma_0}(A_0, C_0(t_0)) \alpha_{\nu_2}^{(1)} \right. \\ & \left. + \sum_{\substack{\nu_1, \nu_2 \in \mathbb{Z} \\ \nu_1 + \nu_2 = 0}} \sum_{\substack{\nu_0, \sigma_0 \in \mathbb{Z} \\ \nu_0 p + \sigma_0 q = \nu_1}} e^{i\sigma_0 t_0} \partial_2 G_{\nu_0, \sigma_0}(A_0, C_0(t_0)) A_{\nu_2}^{(1)} \right) \equiv C_1(t_0), \end{aligned} \quad (4.14)$$

we can express  $C^{(1)}$  in terms of the quantities in (4.12).

In order to study the convergence of the series it is convenient to express all quantities in terms of trees. The strategy is very simple: one iterates the relations (4.6), (4.7) and (4.9), which express the coefficients of order  $k$  in terms of the coefficients of lower order, until we are left only with the coefficients of first order, for which the explicit expressions (4.12), (4.13) and (4.14) are at our disposal.

Trees are defined in the standard way. We briefly recall the basic notations, by referring to [15] for an introductory review and further details, and also to [16, 20] for a discussion in similar contexts.

A tree  $\theta$  is defined as a partially ordered set of points, connected by oriented *lines*. The lines are consistently oriented toward a unique point  $\mathfrak{r}$  called the *root*. The root admits only one entering line called the *root line*. All points except the root are called *nodes*. Denote with  $V(\theta)$  and  $L(\theta)$  the set of nodes and lines in  $\theta$ , respectively, and with  $|L(\theta)|$  and  $|V(\theta)|$  the number of lines and nodes of  $\theta$ , respectively.

If a line  $\ell$  connects two points  $\mathfrak{v}_1, \mathfrak{v}_2$  and is oriented from  $\mathfrak{v}_2$  to  $\mathfrak{v}_1$ , we say that  $\mathfrak{v}_2 \prec \mathfrak{v}_1$  and we shall write  $\ell_{\mathfrak{v}_2} = \ell$ . We shall say also that  $\ell$  exits  $\mathfrak{v}_2$  and enters  $\mathfrak{v}_1$ . It can be convenient to imagine that the line  $\ell$  carries an arrow pointing toward the node  $\mathfrak{v}_1$ : the arrow will be thought of as superimposed on the line itself.

More generally we write  $\mathfrak{v}_2 \prec \mathfrak{v}_1$  if  $\mathfrak{v}_1$  is on the path of lines connecting  $\mathfrak{v}_2$  to the root: hence the orientation of the lines is opposite to the partial ordering relation  $\prec$ . Along the path from  $\mathfrak{v}_2$  to  $\mathfrak{v}_1$  all arrows point toward  $\mathfrak{v}_1$ . In particular all arrows point toward the root.

Each line  $\ell$  carries a pair of labels  $(h_\ell, \delta_\ell)$ , with  $h_\ell \in \{\alpha, A, C\}$  and  $\delta_\ell \in \{1, 2\}$  such that  $\delta_\ell = 1$  for  $h_\ell \neq \alpha$ . We call  $h_\ell$  and  $\delta_\ell$  the *component label* and the *degree label* of the line  $\ell$ , respectively. Given a node  $\mathfrak{v}$  call  $r_{\mathfrak{v}1}, r_{\mathfrak{v}2}$ , and  $r_{\mathfrak{v}3}$  the number of lines entering  $\mathfrak{v}$  carrying a component label  $h = \alpha$ ,  $h = A$ , and  $h = C$ , respectively. Hence, the values of  $r_{\mathfrak{v}1}, r_{\mathfrak{v}2}, r_{\mathfrak{v}3}$  are uniquely determined by the component labels of the lines entering  $\mathfrak{v}$ .

We associate with each node  $\mathfrak{v}$  two *mode labels*  $\nu_{\mathfrak{v}}, \sigma_{\mathfrak{v}} \in \mathbb{Z}$  and we also set for convenience  $\delta_{\mathfrak{v}} = \delta_{\ell_{\mathfrak{v}}}$ . With each line  $\ell$  we associate a further label  $\nu_\ell \in \mathbb{Z}$ , called the *momentum* of the line, such that

$$\nu_\ell = \nu_{\ell_{\mathfrak{v}}} = \sum_{\substack{\mathfrak{w} \in V(\theta) \\ \mathfrak{w} \preceq \mathfrak{v}}} (\nu_{\mathfrak{w}} + \sigma_{\mathfrak{w}}), \quad (4.15)$$

with the constraints that  $\nu_\ell = 0$  if  $h_\ell = C$  and  $\nu_\ell \neq 0$  if  $h_\ell = \alpha$ . The relation (4.15) expresses a conservation law at each node: the momentum of the line exiting  $\mathfrak{v}$  is the sum of the momenta of the lines entering  $\mathfrak{v}$  plus the mode labels of the node  $\mathfrak{v}$  itself. Note that the momentum “flows” through each line in the sense of the arrow superimposed on the line.

The trees with all the labels listed above are called *labelled trees*. Then given a labelled tree  $\theta$  we

associate with each line  $\ell$  a *propagator*

$$g_\ell = \begin{cases} \frac{\omega'(A_0)^{\delta_\ell-1}}{(i\omega\nu_\ell)^{\delta_\ell}}, & h_\ell = \alpha, A, \quad \nu_\ell \neq 0, \\ -\frac{1}{\omega'(A_0)}, & h_\ell = A, \quad \nu_\ell = 0, \\ -\frac{1}{D(t_0)}, & h_\ell = C, \quad \nu_\ell = 0, \end{cases} \quad (4.16)$$

and with each node  $\mathbf{v}$  a *node factor*

$$N_{\mathbf{v}} = \begin{cases} \frac{(i\nu_0)^{r_{\mathbf{v}1}} \partial_2^{r_{\mathbf{v}2}} \partial_3^{r_{\mathbf{v}3}}}{r_{\mathbf{v}1}! r_{\mathbf{v}2}! r_{\mathbf{v}3}!} e^{i\sigma_{\mathbf{v}} t_0} F_{\nu_{\mathbf{v}}}(A_0, C_0(t_0)), & h_{\mathbf{v}} = \alpha, \quad \delta_{\mathbf{v}} = 1, \\ \frac{(i\nu_0)^{r_{\mathbf{v}1}} \partial_2^{r_{\mathbf{v}2}} \partial_3^{r_{\mathbf{v}3}}}{r_{\mathbf{v}1}! r_{\mathbf{v}2}! r_{\mathbf{v}3}!} e^{i\sigma_{\mathbf{v}} t_0} G_{\nu_{\mathbf{v}}}(A_0, C_0(t_0)), & h_{\mathbf{v}} = \alpha, \quad \delta_{\mathbf{v}} = 2, \\ \frac{(i\nu_0)^{r_{\mathbf{v}1}} \partial_2^{r_{\mathbf{v}2}} \partial_3^{r_{\mathbf{v}3}}}{r_{\mathbf{v}1}! r_{\mathbf{v}2}! r_{\mathbf{v}3}!} e^{i\sigma_{\mathbf{v}} t_0} G_{\nu_{\mathbf{v}}}(A_0, C_0(t_0)), & h_{\mathbf{v}} = A, \quad \delta_{\mathbf{v}} = 1, \\ \frac{(i\nu_0)^{r_{\mathbf{v}1}} \partial_2^{r_{\mathbf{v}2}} \partial_3^{r_{\mathbf{v}3}}}{r_{\mathbf{v}1}! r_{\mathbf{v}2}! r_{\mathbf{v}3}!} e^{i\sigma_{\mathbf{v}} t_0} G_{\nu_{\mathbf{v}}}(A_0, C_0(t_0)), & h_{\mathbf{v}} = C, \quad \delta_{\mathbf{v}} = 1, \end{cases} \quad (4.17)$$

with the constraint that when  $h_{\mathbf{v}} = C$  (and  $\delta_{\mathbf{v}} = 1$ ) one has either  $r_{\mathbf{v}3} \geq 2$  or  $r_{\mathbf{v}1} + r_{\mathbf{v}2} \geq 1$ . This constraint reflects the condition  $*$  in (4.10).

Finally we define the *value* of a tree  $\theta$  the number

$$\text{Val}(\theta) = \left( \prod_{\ell \in L(\theta)} g_\ell \right) \left( \prod_{\mathbf{v} \in V(\theta)} N_{\mathbf{v}} \right), \quad (4.18)$$

which is a well-defined quantity: indeed all propagators and node factors are bounded quantities.

Call the *order* of the tree  $\theta$  the number

$$k(\theta) = \{ \ell \in L(\theta) : h_\ell \neq C \}, \quad (4.19)$$

the *total momentum* of  $\theta$  the momentum  $\nu(\theta)$  of the root line, and the *total component label* of  $\theta$  the component label  $h(\theta)$  associated to the root line. The number of nodes (and lines) of any tree  $\theta$  is related to its order  $k(\theta)$  as follows.

**Lemma 1** *For any tree  $\theta$  one has  $|L(\theta)| = |V(\theta)| \leq 2k(\theta)$ .*

*Proof.* The equality  $|L(\theta)| = |V(\theta)|$  is obvious by construction. We prove by induction on  $k$  the bounds

$$|V(\theta)| \leq \begin{cases} 3k(\theta) - 2, & h(\theta) = \alpha, A, \\ 3k(\theta) - 1, & h(\theta) = C. \end{cases} \quad (4.20)$$

For  $k = 1$  the bound (4.20) is trivially satisfied, as a direct check shows: simply compare (4.12) to (4.14) with the definition of trees in that case. Assume that the bound holds for all  $k' < k$ , and let us show that then it holds also for  $k$ . Call  $\ell_0$  the root line of  $\theta$  and  $\mathbf{v}_0$  the node which the root line exits. Call  $r_1$ ,

$r_2$ , and  $r_3$  the number of lines entering  $\mathbf{v}_0$  with component labels  $\alpha$ ,  $A$ , and  $C$ , respectively, and denote with  $\theta_1, \dots, \theta_{r_1+r_2+r_3}$  the subtrees which have those lines as root lines. Then

$$|V(\theta)| = 1 + \sum_{r=j}^{r_1+r_2+r_3} |V(\theta_j)|. \quad (4.21)$$

Then if  $\ell_0$  has component label  $h_{\ell_0} \in \{\alpha, A\}$  we have

$$|V(\theta)| \leq 1 + 3(k-1) - r_3 - 2(r_1 + r_2) \leq 3k - 3 < 3k - 2, \quad (4.22)$$

by the inductive hypothesis and by the fact that  $k(\theta_1) + \dots + k(\theta_{r_1+r_2+r_3}) = k-1$ , whereas if  $\ell_0$  has component label  $h_{\ell_0} = C$  we have

$$|V(\theta)| \leq 1 + 3k - r_3 - 2(r_1 + r_2) \leq 3k - 1, \quad (4.23)$$

by the inductive hypothesis, by the fact that  $k(\theta_1) + \dots + k(\theta_{r_1+r_2+r_3}) = k$ , and by the constraint that either  $r_3 \geq 2$  or  $r_1 + r_2 \geq 1$  in such a case — cf. the comment after (4.17). Therefore the assertion is proved.  $\blacksquare$

Define  $\Theta_{k,\nu,h}$  the set of all trees of order  $k(\theta) = k$ , total momentum  $\nu(\theta) = \nu$ , and total component label  $h(\theta) = h$ . By collecting together all the definitions given above, one obtains that the Fourier coefficients  $\alpha_\nu^{(k)}$  and  $A_\nu^{(k)}$  and the constants  $C_k$  can be written for all  $k \geq 1$  in terms of trees as

$$\begin{aligned} \alpha_\nu^{(k)} &= \sum_{\theta \in \Theta_{k,\nu,\alpha}} \text{Val}(\theta), & \nu \neq 0, & \quad \alpha_0^{(k)} = 0 \\ A_\nu^{(k)} &= \sum_{\theta \in \Theta_{k,\nu,A}} \text{Val}(\theta), & C^{(k)} &= \sum_{\theta \in \Theta_{k,0,C}} \text{Val}(\theta). \end{aligned} \quad (4.24)$$

The proof of (4.24) can be performed by induction; cf. [15] for details.

The number of unlabelled trees of order  $k$  is bounded by the number of random walks of  $2k$  steps, hence by  $2^{2k}$  [23]. The sum over all labels except the mode labels and the momenta is bounded again by a constant to the power  $k$  — simply because all such labels can assume only a finite number of values. Finally the sum over the mode labels — which uniquely determine the momenta through the relation (4.15) — can be performed by using for each node half the exponential decay factor  $e^{-\kappa(|\nu_v| + |\sigma_v|)}$  provided by the bounds (4.4). The conclusion is that we obtain eventually the bounds

$$\left| \alpha_\nu^{(k)} \right| \leq B_1 B_2^k e^{-\kappa|\nu|/2}, \quad \left| A_\nu^{(k)} \right| \leq B_1 B_2^k e^{-\kappa|\nu|/2}, \quad \left| C^{(k)} \right| \leq B_1 B_2^k, \quad (4.25)$$

for suitable constants  $B_1$  and  $B_2$ . This proves the convergence of the series (3.1) and (3.2) for  $|\varepsilon| < \varepsilon_0$ , with  $\varepsilon_0$  small enough. Note that with respect to [15] here the analysis is much easier as there is no small divisors problem.

The construction described above provides also a useful algorithm which can be implemented numerically in order to compute the solution to any prescribed accuracy (provided  $\varepsilon$  is small enough).

Now, we come back to the problem of determining the boundary of the set in the plane  $(\varepsilon, \gamma)$ , with  $\gamma = \varepsilon C$ , in which there are subharmonic solutions of order  $q/p$ .

We have to find the solutions of (4.1), that is, solve the equation

$$0 = \frac{\partial}{\partial t_0} C(\varepsilon, t_0) = C'_0(t_0) + \varepsilon C'_1(t_0) + \varepsilon^2 C'_2(t_0) + \dots, \quad (4.26)$$

where  $C'_k(t_0) = dC_k(t_0)/dt_0$ .

The function  $t_0 \rightarrow C(\varepsilon, t_0)$  is analytic in  $t_0$  for all  $|\varepsilon| < \varepsilon_0$  (for which it is defined and analytic in  $\varepsilon$ ), so that for fixed  $\varepsilon$  the equation (4.26) can always be solved. It has at least the two solutions  $t_0 = \tau_1(\varepsilon)$  and  $t_0 = \tau_2(\varepsilon)$  corresponding to the absolute minimum and to the absolute maximum, respectively, of the function  $C(\varepsilon, t_0)$ . In general these solutions are not smooth in  $\varepsilon$ . This proves Theorem 1.

Suppose now that at the value  $t_0$  such that  $C'_0(t_0) = 0$  one has furthermore  $C''_0(t_0) \neq 0$ . In that case, if  $\tau_0 = \tau_0(\varepsilon)$  is a solution of (4.26) —  $\tau_0$  is a point of minimum or maximum for  $C(\varepsilon, t_0)$  — then  $\tau_0$  must be analytically close to  $t_0$ . Hence  $\varepsilon \rightarrow \tau_0(\varepsilon)$  is an analytic function of  $\varepsilon$ , so that also  $\varepsilon \rightarrow C_1(\varepsilon)$  and  $\varepsilon \rightarrow C_2(\varepsilon)$  are smooth (in fact analytic) in  $\varepsilon$ . Therefore also Theorem 2 follows.

The last observation suggests how to extend Theorem 2 to obtain smooth bifurcation curves when Hypothesis 3 fails to be satisfied.

**Hypothesis 4** *There exists  $k \geq 1$  such that the functions  $C_p(t_0)$  are identically constant in  $t_0$  for all  $p = 0, \dots, k-1$ . If  $t_m$  and  $t_M$  are the values in  $[0, 2\pi)$  for which the function  $C_k(t_0)$  attains its minimum and its maximum, respectively, then  $C''_k(t_m)C''_k(t_M) \neq 0$ .*

The following result extends Theorem 2, as it deals with the case in which the subharmonic Melnikov function does not depend explicitly on  $t_0$ , that is  $C'_0(t_0) \equiv 0$ .

**Theorem 3** *Consider the system (2.1) and assume that Hypotheses 1, 2 and 4 hold for the resonant torus with energy  $A_0$  such that  $\omega(A_0) = p/q$ . There exist  $\varepsilon_0 > 0$  and two functions  $\tilde{\gamma}_1(\varepsilon)$  and  $\tilde{\gamma}_2(\varepsilon)$ , analytic for  $|\varepsilon| < \varepsilon_0$ , with  $\tilde{\gamma}_1(0) = \tilde{\gamma}_2(0)$  and  $\tilde{\gamma}_1(\varepsilon) \neq \tilde{\gamma}_2(\varepsilon)$  for all  $\varepsilon \neq 0$ , such that the two functions*

$$\gamma_1(\varepsilon) = \begin{cases} \max\{\tilde{\gamma}_1(\varepsilon), \tilde{\gamma}_2(\varepsilon)\}, & \varepsilon > 0, \\ \min\{\tilde{\gamma}_1(\varepsilon), \tilde{\gamma}_2(\varepsilon)\}, & \varepsilon < 0, \end{cases} \quad \gamma_2(\varepsilon) = \begin{cases} \min\{\tilde{\gamma}_1(\varepsilon), \tilde{\gamma}_2(\varepsilon)\}, & \varepsilon > 0, \\ \max\{\tilde{\gamma}_1(\varepsilon), \tilde{\gamma}_2(\varepsilon)\}, & \varepsilon < 0, \end{cases} \quad (4.27)$$

*have the same tangent lines at the origin, and (2.1) has at least one subharmonic solution of order  $q/p$  for  $\gamma_2(\varepsilon) \leq \varepsilon C \leq \gamma_1(\varepsilon)$  when  $\varepsilon \in (0, \varepsilon_0)$  and for  $\gamma_1(\varepsilon) \leq \varepsilon C \leq \gamma_2(\varepsilon)$  when  $\varepsilon \in (-\varepsilon_0, 0)$ .*

The proof follows the same lines as that of Theorem 2. The only difference is that up to order  $k-1$  the initial phase is left undetermined. In fact to first order one has  $M(t_0, C) = M(C) = 0$  which fixes  $C = C_0$  (by Hypothesis 2), while to orders  $k' = 2, \dots, k-1$  the constants  $C_k$  are fixed and are independent of  $t_0$  by Hypothesis 4. Then we can write  $C(\varepsilon, t_0) = \mathfrak{C}_1(\varepsilon) + \mathfrak{C}_2(\varepsilon, t_0)$ , with  $\mathfrak{C}_1(\varepsilon) = C_0 + \varepsilon C_1 + \dots + \varepsilon^{k-1} C_{k-1}$  and  $\mathfrak{C}_2(\varepsilon, t_0) = \varepsilon^k (C_k(t_0) + O(\varepsilon))$ , and from order  $k$  on the constants  $C_k$  are fixed as functions of  $t_0$ . Moreover equation (4.26) reduces to  $0 = C'_k(t_0) + \varepsilon C'_{k+1}(t_0) + \dots$ . Therefore we can reason as in the previous case ( $k = 0$ ) and we find that  $C_k(t_0)$  has at least two stationary points  $t_0 = t_1$  and  $t_0 = t_2$ , corresponding to the minimum point and to the maximum point, respectively. By Hypothesis 4 also  $\mathfrak{C}_2(\varepsilon, t_0)$  has two stationary points at  $\tau_1(\varepsilon) = t_1 + O(\varepsilon)$  and  $\tau_2(\varepsilon) = t_2 + O(\varepsilon)$ , with  $\tau_1(\varepsilon)$  and  $\tau_2(\varepsilon)$  analytic in  $\varepsilon$  for  $\varepsilon$  small enough. Then we can define  $\tilde{\gamma}_1(\varepsilon) = C(\varepsilon, \tau_1(\varepsilon))$  and  $\tilde{\gamma}_2(\varepsilon) = C(\varepsilon, \tau_2(\varepsilon))$ : by construction, both  $\tilde{\gamma}_1(\varepsilon)$  and  $\tilde{\gamma}_2(\varepsilon)$  are analytic in  $\varepsilon$  for  $\varepsilon$  small enough. If we define  $\gamma_1(\varepsilon)$  and  $\gamma_2(\varepsilon)$  according to (4.27) then the proof of the theorem is achieved.

Note that in this case the definition (4.27) coincides with the general definition (4.1) for the bifurcation curves. Furthermore, if we assume Hypothesis 3 instead of Hypothesis 4, then one has  $\tilde{\gamma}_1(\varepsilon) = \gamma_1(\varepsilon)$  and  $\tilde{\gamma}_2(\varepsilon) = \gamma_2(\varepsilon)$ , so that also  $\gamma_1(\varepsilon)$  and  $\gamma_2(\varepsilon)$  are analytic, as stated in Theorem 2.

Finally we note that if the functions  $C_k(t_0)$  are identically constant in  $t_0$  for all  $k \in \mathbb{Z}_+$  then one has  $C(\varepsilon, t_0) = C(\varepsilon)$ . In this case the two curves  $\gamma_1(\varepsilon)$  and  $\gamma_2(\varepsilon)$  coincide, and all values of  $t_0$  are allowed. This means that the whole manifold corresponding to the resonant torus persists. On the other hand the parameter  $C$  must be fixed in a very precise way, as a function of  $\varepsilon$ , and any small deviation from that value destroys the torus. This result can be compared with [9, 10], where a similar situation is discussed.

For  $(\varepsilon, \gamma)$  inside the set of existence of subharmonic solutions one can investigate how many of them exist. For  $p = 1$  the initial phase  $t_0$  varies in the interval  $[0, 2\pi q]$ , where  $T_0 = 2\pi q$  is the period of the

unperturbed periodic solution. The function  $C(\varepsilon, t_0)$  has period  $2\pi$  in  $t_0$ , so that it is repeated  $q$  times in the interval  $[0, 2\pi q]$ . Hence for any fixed value  $|\varepsilon| < \varepsilon_0$  and any  $C$  strictly between the maximum and the minimum value attained by the function  $t_0 \rightarrow C(\varepsilon, t_0)$  there are at least  $2q$  values  $t_i$ ,  $i = 2, \dots, 2q$ , such that  $C = C(\varepsilon, t_i)$ . If  $C$  coincides with either its maximum or its minimum then there are at least  $q$  values  $t_i$ ,  $i = 2, \dots, q$ , such that  $C = C(\varepsilon, t_i)$ . Therefore we can conclude that, for  $p = 1$ , inside the set of existence of subharmonic solutions there are at least  $2q$  such solutions, as found in [11], while on the boundary of that set there are  $q$  of them.

We can summarise the discussion above in the following statement.

**Theorem 4** *Under the same assumptions of Theorem 1 assume  $p = 1$ . Take  $|\varepsilon| < \varepsilon_0$ , and for such values of  $\varepsilon$  let  $\varepsilon \rightarrow \gamma_1(\varepsilon)$  and  $\varepsilon \rightarrow \gamma_2(\varepsilon)$  be the two bifurcation curves whose existence is assured by Theorem 1. For  $\min\{\gamma_1(\varepsilon), \gamma_2(\varepsilon)\} < \gamma < \max\{\gamma_1(\varepsilon), \gamma_2(\varepsilon)\}$  there at least  $2q$  subharmonic solutions of order  $q$ . If either  $\gamma = \gamma_1(\varepsilon)$  or  $\gamma = \gamma_2(\varepsilon)$  one has at least  $q$  subharmonic solutions of order  $q$ .*

Theorem 4 should be compared with Theorem 2.1 in [11].

## 5 Application to dissipative systems with forcing

Let us consider a one-dimensional system, subject to a conservative force  $g(x)$ , in the presence of dissipation and of a periodic forcing. If the periodic forcing and the dissipation coefficient are both small we can write the equations for the system as

$$\ddot{x} + g(x) + \gamma \dot{x} = \varepsilon f(x, t), \quad \gamma = \varepsilon C, \quad (5.1)$$

where  $\varepsilon f(x, t)$  is the forcing of period  $2\pi$  and  $C$  is a parameter. Assume that both  $g$  and  $f$  are analytic in their arguments. If  $f$  depends only on  $t$ , equation (5.1) reduces to the equation studied in [25].

Let us assume that the unperturbed system ( $\varepsilon = 0$ ) is Liouville-integrable and anysochronous. This means that, in action-angle variables, the equations (5.1) can be written in the form (2.1), and, furthermore, that Hypothesis 1 is satisfied.

We define the subharmonic Melnikov function in terms of the action-angle variable as in (2.2). To check that Hypothesis 2 is also satisfied we use the following result.

**Lemma 2** *The subharmonic Melnikov function is invariant under a transformation of coordinates.*

*Proof.* Consider a system of differential equations in  $\mathbb{R}^2$

$$\dot{x} = f(x) + \varepsilon g(x, t), \quad (5.2)$$

and define the subharmonic Melnikov function [29, 24, 11] for a subharmonic solution  $x_0(t)$  of period  $T$  as

$$M(t_0) = \frac{1}{T} \int_0^T dt (f_1(x_0(t)) g_2(x_0(t)) - f_2(x_0(t)) g_1(x_0(t))). \quad (5.3)$$

Take the transformation of coordinates  $\xi \rightarrow x = h(\xi)$ . In the new coordinates the system reads

$$\dot{\xi} = \phi(\xi) + \varepsilon \gamma(\xi, t), \quad (5.4)$$

where  $\phi(\xi) = \partial h^{-1}(h(\xi)) f(h(\xi))$  and  $\gamma(\xi) = \partial h^{-1}(h(\xi)) g(h(\xi))$ , and the subharmonic Melnikov function becomes

$$\mathcal{M}(t_0) = \frac{1}{T} \int_0^T dt (\phi_1(\xi_0(t)) \gamma_2(\xi_0(t)) - \phi_2(\xi_0(t)) \gamma_1(\xi_0(t))), \quad (5.5)$$

where  $\xi_0(t)$  is the subharmonic solution expressed in the new variables.

By noting that

$$\partial h^{-1}(h(\xi)) = (\partial h(\xi))^{-1} = \frac{1}{J} \begin{pmatrix} \partial_2 h_2(\xi) & -\partial_2 h_1(\xi) \\ -\partial_1 h_2(\xi) & \partial_1 h_1(\xi) \end{pmatrix}, \quad (5.6)$$

where  $J = \det \partial h = \partial_1 h_1 \partial_2 h_2 - \partial_1 h_2 \partial_2 h_1$  is the Jacobian of the transformation, one obtains

$$\begin{aligned} \mathcal{M}(t_0) &= \frac{1}{T} \int_0^T dt \frac{1}{J} \left( (\partial_2 h_2 f_1 - \partial_2 h_1 f_2) (-\partial_1 h_2 g_1 + \partial_1 h_1 g_2) - \right. \\ &\quad \left. (-\partial_1 h_2 f_1 + \partial_1 h_1 f_2) (\partial_2 h_2 g_1 - \partial_2 h_1 g_2) \right) \\ &= \frac{1}{T} \int_0^T dt \frac{1}{J} (\partial_1 h_1 \partial_2 h_2 - \partial_1 h_2 \partial_2 h_1) (f_1 g_2 - f_2 g_1), \end{aligned} \quad (5.7)$$

where the function  $h$  is computed in  $\xi_0(t)$  and the functions  $f, g$  are computed in  $x_0(t) = h(\xi_0(t))$ . Hence (5.3) yields  $\mathcal{M}(t_0) = M(t_0)$ , so that the assertion follows.  $\blacksquare$

This means that we can compute the subharmonic Melnikov function for the system (5.1) in the coordinates  $(x, y) = (x, \dot{x})$ . In that case the unperturbed vector field is  $(y, -g(x))$  and the perturbation reads  $(0, -\varepsilon C y + \varepsilon f(x, t))$ , so that the subharmonic Melnikov function becomes

$$M(t_0, C) = \frac{1}{T} \int_0^T dt y_0(t) (-C y_0(t) + f(x_0(t), t + t_0)) = -C \langle y_0^2 \rangle + \langle y_0 f(x_0(\cdot), \cdot + t_0) \rangle. \quad (5.8)$$

Therefore the subharmonic Melnikov function vanishes provided  $C = C_0(t_0)$ , where  $C_0(t_0) = (\langle y_0^2 \rangle)^{-1} \langle y_0 f(x_0(\cdot), \cdot + t_0) \rangle$ , which is well-defined because  $\langle y_0^2 \rangle > 0$ . Moreover one has  $\partial M(t_0, C) / \partial C = -\langle y_0^2 \rangle \neq 0$ . Therefore Hypothesis 2 is also satisfied, and Theorem 2 applies to the system (5.1).

We can state our result as follows.

**Theorem 5** *Consider the system (5.1) and assume that Hypothesis 1 holds for the invariant torus with energy  $A_0$  such that  $\omega(A_0) = p/q$ . There exist  $\varepsilon_0 > 0$  and two continuous functions  $\gamma_1(\varepsilon)$  and  $\gamma_2(\varepsilon)$ , with  $\gamma_1(0) = \gamma_2(0)$ ,  $\gamma_1(\varepsilon) \geq \gamma_2(\varepsilon)$  for  $\varepsilon \geq 0$  and  $\gamma_1(\varepsilon) \leq \gamma_2(\varepsilon)$  for  $\varepsilon \leq 0$ , such that (2.1) has at least one subharmonic solution of period  $2\pi p$  for  $\gamma_2(\varepsilon) \leq \varepsilon C \leq \gamma_1(\varepsilon)$  when  $\varepsilon \in (0, \varepsilon_0)$  and for  $\gamma_1(\varepsilon) \leq \varepsilon C \leq \gamma_2(\varepsilon)$  when  $\varepsilon \in (-\varepsilon_0, 0)$ .*

Of course Theorem 5 is a corollary of Theorem 1. It should be compared with Corollary 2.3 in [11] (cf. also [25]). Our result is stronger as it requires, in Chow and Hale's notations, only Hypothesis  $(H_1)$ , which corresponds to our Hypothesis 1. If one assumes also Hypothesis  $(H_4)$  of [11], which corresponds to our hypothesis 3, then Theorem 2 applies, and the result of [11] is recovered.

One expects that, in the case of system (5.1), the two bifurcation curves  $\gamma_1(\varepsilon)$  and  $\gamma_2(\varepsilon)$  contain the real axis, that is  $\min\{\gamma_1(\varepsilon), \gamma_2(\varepsilon)\} \leq 0 \leq \max\{\gamma_1(\varepsilon), \gamma_2(\varepsilon)\}$ . Indeed for  $\gamma = 0$  the equation (5.1) describes a quasi-integrable Hamiltonian system, and existence of periodic solutions is well known in this case, at least under some non-degeneracy condition on the unperturbed system, such as Hypothesis 1. If  $C_0(t_0)$  is not zero then it is easy to check that the set of existence of subharmonic solutions includes the real axis. Indeed, this follows from the following result.

**Lemma 3** *The function  $C_0(t_0)$  has zero mean.*

*Proof.* Call

$$F(x_0(t)) = \int_0^{2\pi} \frac{dt_0}{2\pi} f(x_0(t), t + t_0) = \int_0^{2\pi} \frac{dt_0}{2\pi} f(x_0(t), t_0). \quad (5.9)$$

By (5.8) the mean (with respect to  $t_0$ ) of  $C_0(t_0)$  is

$$\begin{aligned} \int_0^{2\pi} \frac{dt_0}{2\pi} C_0(t_0) &= \frac{1}{\langle y_0^2 \rangle} \int_0^{2\pi} \frac{dt_0}{2\pi} \int_0^T \frac{dt}{T} \dot{x}_0(t) f(x_0(t), t + t_0) \\ &= \int_0^T \frac{dt}{T} \dot{x}_0(t) F(x_0(t)), \end{aligned} \quad (5.10)$$

which vanishes, as the integrand can be written as a total derivative with respect to  $t$ .  $\blacksquare$

In particular Lemma 3 implies that if  $C_0(t_0)$  is not identically constant then its maximum is strictly positive and its minimum is strictly negative, hence  $\max\{\gamma_1(\varepsilon), \gamma_2(\varepsilon)\} > 0$  and  $\min\{\gamma_1(\varepsilon), \gamma_2(\varepsilon)\} < 0$ .

To extend the same result to the case in which the functions  $C_{k'}(t_0)$  are identically constant in  $t_0$  for all  $k' \leq k-1$ , with  $k \geq 1$  arbitrarily high, is more delicate, and it requires some work. One can reason as follows.

**Lemma 4** *Assume that for some  $\bar{k} \in \mathbb{Z}_+$  the coefficients  $C_{k'}(t_0)$  vanish identically for all  $k' = 0, \dots, \bar{k} - 1$ . Then  $C_{\bar{k}}(t_0)$  has zero mean in  $t_0$ .*

*Proof.* Write the system (5.1) in action-angle variables. Then there exists a Hamiltonian function  $H(\alpha, A, t, \varepsilon) = H_0(A) + \varepsilon H_1(\alpha, A, t)$  such that  $\omega(A) = \partial_A H_0(A)$  and

$$\begin{cases} \dot{\alpha} = \omega(A) + \varepsilon \partial_A H_1(\alpha, A, C, t) + \varepsilon C \Phi(\alpha, A), \\ \dot{A} = -\varepsilon \partial_\alpha H_1(\alpha, A, C, t) + \varepsilon C \Psi(\alpha, A), \end{cases} \quad (5.11)$$

where  $\Phi = -y \partial \alpha / \partial y$  and  $\Psi = y \partial A / \partial y$ . Then (4.6) become

$$\begin{aligned} \alpha_\nu^{(k)} &= \frac{1}{i\omega\nu} \left( \partial_A H_1^{(k-1)} \right)_\nu + \omega'(A_0) \frac{1}{(i\omega\nu)^2} \left( -\partial_\alpha H_1^{(k-1)} \right)_\nu + (C\Phi)_\nu^{(k-1)}, \\ A_\nu^{(k)} &= \frac{1}{i\omega\nu} \left( -\partial_\alpha H_1^{(k-1)} \right)_\nu + (C\Psi)_\nu^{(k-1)}, \end{aligned} \quad (5.12)$$

for all  $k \in \mathbb{N}$  and all  $\nu \neq 0$ . Moreover (4.9) reads

$$\sum_{k'=0}^k C_{k'} \Psi_0^{(k')} + \bar{\Gamma}_0^{(k)} = 0, \quad \bar{\Gamma}_0^{(k)} = \left( -\partial_\alpha H_1^{(k-1)} \right)_0, \quad (5.13)$$

which, for  $k = \bar{k}$ , gives  $\Gamma_0^{(\bar{k})} = \bar{\Gamma}_0^{(\bar{k})}$  and  $C_{\bar{k}} \Psi_0^{(0)} + \bar{\Gamma}_0^{(\bar{k})} = 0$  because  $C_1 = \dots = C_{\bar{k}-1} = 0$  by assumption. Moreover  $\Psi^{(0)} = -\langle y_0^2 \rangle \neq 0$ , by Lemma 2 and Hypothesis 2.

Therefore  $C_k = C^{(k)}$ , with  $C^{(k)}$  given by the sum (4.24) of tree values. We can split the set  $\Theta_{k,0,C}$  into the union of disjoint families  $\mathcal{F}$  as follows. Given a tree  $\theta \in \Theta_{k,0,C}$  call  $\mathbf{v}_0$  the node which is connected to the root through the root line, and define  $V_0(\theta)$  as the subset of nodes  $\mathbf{v} \in V(\theta)$  such that all the lines  $\ell$  along the path connecting  $\mathbf{v}$  to  $\mathbf{v}_0$  have  $\nu_\ell \neq 0$ . Then define  $\mathcal{F} = \mathcal{F}(\theta)$  as the set of trees obtained from  $\theta$  by “shifting” the root line to any node in  $V_0(\theta)$ , i.e. by attaching the root line to any node  $\mathbf{v} \in V_0(\theta)$ . Of course, as a consequence of the shift of the root line from  $\mathbf{v}_0$  to  $\mathbf{v}$ , the arrows of all lines along the path between the two nodes are reversed. If one recalls the diagrammatic rules introduced in Section 4 to associate with any tree  $\theta$  a value  $\text{Val}(\theta)$ , this means that all lines with labels  $(h, \delta) = (\alpha, 1)$  are transformed into lines with labels  $(h, \delta) = (A, 1)$ . Moreover the momenta of all such lines change sign. The latter property can be seen as follows. The momentum is defined as the sum of all mode labels of the nodes preceding the lines — cf. (4.15) — and the sum of all the mode labels is zero for any tree  $\theta \in \Theta_{k,0,C}$ : then, when the arrow of a line  $\ell$  is reversed the nodes preceding  $\ell$  become the nodes following

$\ell$  and vice versa, so that  $\nu_\ell$  becomes  $-\nu_\ell$ . Hence the propagators of the lines  $\ell$  with  $\delta_\ell = 1$  change sign, whereas the propagators of the lines  $\ell$  with  $\delta_\ell = 2$  are left unchanged. As a consequence, for each tree  $\theta' \in \mathcal{F}(\theta)$  we can write  $\text{Val}(\theta) = i\nu_{\mathbf{v}} \overline{\text{Val}}(\theta)$ , where  $\mathbf{v}$  is the node  $\mathbf{v} \in V_0(\theta)$  which the root line exits and  $\overline{\text{Val}}(\theta)$  is the same quantity for all  $\theta' \in \mathcal{F}(\theta)$ . Therefore

$$\sum_{\theta' \in \mathcal{F}(\theta)} \text{Val}(\theta) = \overline{\text{Val}}(\theta) \sum_{\mathbf{v} \in V_0(\theta)} i\nu_{\mathbf{v}}. \quad (5.14)$$

Moreover one has

$$\sum_{\mathbf{v} \in V(\theta)} (\nu_{\mathbf{v}} + \sigma_{\mathbf{v}}) = 0 \implies \sum_{\mathbf{v} \in V_0(\theta)} (\nu_{\mathbf{v}} + \sigma_{\mathbf{v}}) = 0 \implies \sum_{\mathbf{v} \in V_0(\theta)} \nu_{\mathbf{v}} = - \sum_{\mathbf{v} \in V_0(\theta)} \sigma_{\mathbf{v}}, \quad (5.15)$$

so that the mean in  $t_0$  of (5.14) gives

$$\int_0^{2\pi} \frac{dt_0}{2\pi} \sum_{\theta' \in \mathcal{F}(\theta)} \text{Val}(\theta) = \int_0^{2\pi} \frac{dt_0}{2\pi} \overline{\text{Val}}(\theta) \sum_{\mathbf{v} \in V_0(\theta)} i\nu_{\mathbf{v}} = - \int_0^{2\pi} \frac{dt_0}{2\pi} \overline{\text{Val}}(\theta) \sum_{\mathbf{v} \in V_0(\theta)} i\sigma_{\mathbf{v}} = 0, \quad (5.16)$$

because the mean is the sum over all labels  $\sigma_{\mathbf{v}} \in V(\theta)$  such that  $\sum_{\mathbf{v} \in V(\theta)} \sigma_{\mathbf{v}} = \sum_{\mathbf{v} \in V_0(\theta)} \sigma_{\mathbf{v}} = 0$ . By using the fact that the set  $\Theta_{k,0,C}$  can be written as a disjoint union of the sets  $\mathcal{F}$ , we obtain that  $\bar{\Gamma}_0^{(\bar{k})}$  has zero mean in  $t_0$ , so that the assertion follows.  $\blacksquare$

**Lemma 5** *Assume that for some  $\bar{k} \in \mathbb{Z}_+$  the coefficients  $C_{k'}(t_0)$  are identically constant for all  $k' = 0, \dots, \bar{k} - 1$ . Then  $C_{k'}(t_0) \equiv 0$  for all  $k' = 0, \dots, \bar{k} - 1$ .*

*Proof.* The proof is by induction. Fix  $0 \leq k < \bar{k}$ , and assume that  $C_{k'}(t_0) \equiv 0$  for all  $k' \leq k - 1$ . Then by Lemma 4 the function  $C_k(t_0)$  has zero mean. Since it is constant by hypothesis then  $C_k(t_0) \equiv 0$ .  $\blacksquare$

Let  $k \in \mathbb{Z}_+$  be such that  $C_{k'}(t_0)$  is identically constant in  $t_0$  for  $k' = 0, \dots, k - 1$  whereas  $C_k(t_0)$  depends explicitly on  $t_0$ . If  $k = 0$  this simply means that  $C_0(t_0)$  depends explicitly on  $t_0$ . By Lemma 5 one has  $C_{k'}(t_0) \equiv 0$  for all  $k' \leq k - 1$ , and by Lemma 4 the function  $C_k(t_0)$  has zero mean in  $t_0$ . Since  $C_k(t_0)$  is not identically constant then  $\sup_{t_0 \in [0, 2\pi)} C_k(t_0) > 0$  and  $\inf_{t_0 \in [0, 2\pi)} C_k(t_0) < 0$ . Furthermore, in such a case  $C(\varepsilon, t_0) = \varepsilon^k (C_k(t_0) + O(\varepsilon))$ , so that also

$$\sup_{t_0 \in [0, 2\pi)} C(\varepsilon, t_0) > 0, \quad \inf_{t_0 \in [0, 2\pi)} C(\varepsilon, t_0) < 0, \quad (5.17)$$

for  $\varepsilon$  small enough. If we recall the definition (4.1) of the bifurcation curves we can formulate the following result.

**Theorem 6** *Under the same assumptions of Theorem 5 let  $\varepsilon \rightarrow \gamma_1(\varepsilon)$  and  $\varepsilon \rightarrow \gamma_2(\varepsilon)$  be the two bifurcation curves whose existence is assured by Theorem 5. One has  $\gamma_1(\varepsilon) \geq 0 \geq \gamma_2(\varepsilon)$  for  $\varepsilon \in (0, \varepsilon_0)$  and  $\gamma_1(\varepsilon) \leq 0 \leq \gamma_2(\varepsilon)$  for  $\varepsilon \in (0, \varepsilon_0)$ .*

As (5.17) shows, if there is  $k \geq 0$  such that  $C_{k'}(t_0) \equiv 0$  for  $k' = 0, \dots, k - 1$  and  $C_k(t_0) \neq 0$ , then one has the strict inequalities  $\gamma_1(\varepsilon) > 0 > \gamma_2(\varepsilon)$  for  $\varepsilon \in (0, \varepsilon_0)$  and  $\gamma_1(\varepsilon) < 0 < \gamma_2(\varepsilon)$  for  $\varepsilon \in (0, \varepsilon_0)$ . On the contrary if all  $C_k$  vanish identically, so that the full function  $C(\varepsilon, t_0)$  has to be zero, then  $\gamma_1(\varepsilon) = \gamma_2(\varepsilon) = 0$ .

Therefore Theorems 5 and 6 show that any one-dimensional anisochronous mechanical system, when perturbed by a periodic forcing and in the presence of dissipation, up to the exceptional cases in which the functions  $C_k(t_0)$  are constant — hence vanish, by Lemma 5 — in  $t_0$  for all  $k \in \mathbb{Z}_+$ , admits subharmonic



solutions of all orders, without any assumption on the perturbation, — a result which does not follow from the analysis of [25, 11].

The case that all the functions  $C_k(t_0)$  are identically constant in  $t_0$  is really exceptional. This can be appreciated by the following argument. If the function  $C(\varepsilon, t_0)$  does not depend on  $t_0$  then not only, by Lemma 5, it must vanish identically, i.e.  $C(\varepsilon, t_0) = C(\varepsilon) \equiv 0$ , but we find also that  $t_0$  is left undetermined. In other words the periodic solution persists for all values of  $t_0$ . This means that if we take the system (5.1) with  $\gamma = 0$ , so that it becomes an autonomous quasi-integrable Hamiltonian system, with no dissipation left, the full resonant torus with frequency  $\omega = p/q$  persists under perturbation. This situation is certainly unlikely — even if not impossible in principle. For instance one can take the system described by the Hamiltonian

$$H(x, y, t) = \frac{1}{2}y^2 + \frac{1}{4}x^4 + \varepsilon f(t) \left( \frac{1}{2}y^2 + \frac{1}{4}x^4 - E \right)^2, \quad (5.18)$$

with  $E$  corresponding to the unperturbed solution  $(x_0(t), y_0(t))$  with frequency  $\omega$ . Then such a solution still satisfies the corresponding Hamilton equations for all values of  $\varepsilon$  and for all values of the initial phase  $t_0$ : that is the full resonant torus with frequency  $\omega$  persists. In particular if  $\omega = p/q$  is rational — so that the frequency of the unperturbed solution becomes commensurate with the frequency 1 of the perturbing potential  $f$ , the corresponding torus is resonant.

It is important to stress that if we look for a subharmonic solution which continues some unperturbed periodic solution with a given period  $T = 2\pi q/p$  it can happen that the corresponding integral  $\langle y_0 f(x_0(\cdot), \cdot + t_0) \rangle$  identically vanishes. In fact, if  $f$  is a trigonometric polynomial (which is often the case in physical applications) this happens for all  $p/q$  but a finite set of values. An explicit example has been considered in [1]. In these cases the subharmonic Melnikov function does not depend on  $t_0$  and it is linear in  $C$ : hence (5.8) can be satisfied only by taking  $C_0(t_0) \equiv 0$ . Then, it becomes essential to go to higher orders of perturbation theory to study for which values of  $C$  a subharmonic solution of order  $q/p$  appears. Again, we refer to [1] for a situation in which one must perform a higher order analysis to explain the numerical findings.

## 6 Conclusions, and final comments

The Melnikov theory [24] considers systems which, in suitable coordinates, can be written as in (2.1), without the parameter  $C$ :

$$\begin{cases} \dot{\alpha} = \omega(A) + \varepsilon F(\alpha, A, t), \\ \dot{A} = \varepsilon G(\alpha, A, t), \end{cases} \quad (6.1)$$

where all notations are as explained after (2.1). Define the subharmonic Melnikov function as

$$M(t_0) = \frac{1}{T} \int_0^T dt G(\alpha_0(t), A_0(t), t + t_0), \quad (6.2)$$

and set  $M'(t_0) = dM(t_0)/dt_0$ .

We can repeat the analysis of formal solvability in Section 3, with some adaptations due to the fact that no extra parameters  $C^{(k)}$  are at our disposal to any perturbation orders.

In particular to first order one needs  $M(t_0) = 0$ , so that  $t_0$  must be a zero for the subharmonic Melnikov function. To higher orders we can write

$$G^{(k)}(\alpha(t), A(t), C, t + t_0) = \partial_1 G(\alpha_0(t), A_0, t + t_0) \bar{\alpha}^{(k)} + \Gamma^{(k)}(\alpha(t), A(t), t + t_0), \quad (6.3)$$

where the function  $\Gamma^{(k)}(\alpha(t), A(t), t + t_0)$  depends on the corrections  $\bar{\alpha}^{(k')}$  to the initial phase, only with  $k' < k$ .

To any perturbation order  $k$  the constant  $\bar{\alpha}^{(k)}$  is left undetermined. Anyway we are no longer free to fix it equal to some arbitrary value, for instance zero, as we no longer have the initial phase  $t_0$  and the constants  $C^{(k)}$  as free parameters. Hence we shall need the corrections  $\bar{\alpha}^{(k)}$  to assure solvability of the equations of motion to any order. This will be possible in the light of the following results.

**Lemma 6** *One has  $\omega(A_0)\langle\partial_1 G(\alpha_0(\cdot), A_0, \cdot + t_0)\rangle = -M'(t_0)$ .*

*Proof.* One has

$$\frac{d}{dt}G(\alpha_0(t), A_0, t + t_0) = \omega(A_0)\partial_1 G(\alpha_0(t), A_0, t + t_0) + \frac{\partial}{\partial t_0}G(\alpha_0(t), A_0, t + t_0), \quad (6.4)$$

where we have used the fact that  $\dot{A}_0(t) = 0$  and  $\dot{\alpha}_0(t) = \omega(A_0)$ . If we integrate (6.4) over a period we obtain

$$0 = \frac{1}{T} \int_0^T dt \frac{d}{dt}G(\alpha_0(t), A_0, t + t_0) = \omega(A_0)\langle\partial_1 G(\alpha_0(\cdot), A_0, \cdot + t_0)\rangle + \frac{\partial}{\partial t_0}\langle G(\alpha_0(\cdot), A_0, \cdot + t_0)\rangle, \quad (6.5)$$

so that

$$\omega(A_0)\partial_1 G(\alpha_0(t), A_0, C_0, t + t_0) = -\frac{\partial}{\partial t_0}\langle G(\alpha_0(t), A_0, t + t_0)\rangle = -M'(t_0). \quad (6.6)$$

Hence the assertion follows. ■

Thus, if we impose the condition that  $t_0$  be a simple zero for the subharmonic Melnikov function we find that in (6.3) the derivative  $\partial_1 G(\alpha_0(t), A_0, t + t_0)$  is different from zero, and this allows us to fix  $\bar{\alpha}^{(k)}$  in such a way as to make the mean of  $G^{(k)}(\alpha(t), A(t), t + t_0)$  vanish. Hence by fixing the constants  $\bar{A}^{(k)}$  as explained in Section 3 and the constants  $\bar{\alpha}^{(k)}$  as stated above we find that a solution in the form of a formal power series in  $\varepsilon$  exists. The convergence of the series, hence the existence of an analytic solution, can be proved by reasoning as in Section 4. We do not repeat the analysis, which would essentially be a word for word copy of what was done in Section 4.

Therefore we have proved the following result — well-known in the literature [24].

**Theorem 7** *Consider a periodic solution with frequency  $\omega = p/q$  for the system (6.1), and assume that  $t_0$  is a simple zero for the subharmonic Melnikov function (6.2) corresponding to such a solution. There exists  $\varepsilon_0 > 0$  such that for  $|\varepsilon| < \varepsilon_0$  the system (6.1) has at least one subharmonic solution of order  $q/p$ .*

However, our analysis permits us to generalise the result above. Define

$$M_0(t_0) = M(t_0), \quad M_k(t_0) = \langle \Gamma^{(k)}(\alpha(\cdot), A(\cdot), \cdot + t_0) \rangle, \quad k \in \mathbb{N}, \quad (6.7)$$

where the notations of (6.3) have been used. Note that if  $M_{k'}(t_0)$  vanishes identically for all  $k' = 0, 1, \dots, k-1$ , then  $M_k(t_0)$  is well-defined. The following result follows.

**Theorem 8** *Consider a periodic solution with frequency  $\omega = p/q$  for the system (6.1). Assume that the functions  $M_{k'}$  are identically zero for all  $k' = 0, 1, \dots, k-1$ , and assume that  $t_0$  is a simple zero for the function  $M_k(t_0)$ . There exists  $\varepsilon_0 > 0$  such that for  $|\varepsilon| < \varepsilon_0$  the system (6.1) has at least one subharmonic solution of order  $q/p$ .*

Of course also the system (2.1) can be studied as illustrated in this section. One simply treats the parameter  $C$  as fixed, and one fixes the initial phase  $t_0$  in such a way that Theorem 7 or Theorem 8 can be applied — of course, provided the corresponding hypotheses are satisfied. This has been done in [1] to study the subharmonic solutions of a forced cubic oscillator in the presence of dissipation.

We also note that, as a particular case of Theorem 8, it can happen that  $M_k(t_0) \equiv 0$  for all  $k \in \mathbb{Z}_+$ . In that case formal solvability of the equations holds to all orders, and the convergence of the series requires no condition on  $t_0$ , and it can be proved by proceeding as in Section 4. In particular in such a case the full resonant torus persists under perturbation. Of course, the vanishing identically of all functions  $M_k$  is a very unlikely situation, and, without any further parameter at our disposal, we can hardly expect this ever to happen. This shows that the persistence of the full torus when the subharmonic Melnikov function is identically zero is a very rare event.

The bifurcation curves studied in this paper concern subharmonic solutions which are analytic in  $\varepsilon$ . In principle our results do not exclude existence of other subharmonic solutions which are not analytic. Indeed, one could wonder whether other periodic solutions with the same period exist for  $\varepsilon \neq 0$ . In the presence of dissipation, it is unlikely that solutions other than the attractive ones found with the method we have used, would be relevant for the dynamics — cf. for instance the problems investigated in [4, 2, 3, 1, 5]. In general the situation can be delicate; for instance when one investigates quasi-periodic solutions corresponding to lower-dimensional tori of quasi-integrable systems, where uniqueness becomes a subtle problem — cf. for instance [22, 12]. Despite this, there are cases in which the problem can be settled — cf. [3, 19].

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